

XIV. *A Method of finding the Value of an infinite Series of decreasing Quantities of a certain Form, when it converges too slowly to be summed in the common Way by the mere Computation and Addition or Subtraction of some of its initial Terms.* By Francis Maſeres, Eſquire, F. R. S. Curſitor Baron of the Exchequer.

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ARTICLE Iſt. Let $a, b, c, d, e, f, g, h, \&c.$ *ad infinitum*, represent a decreasing progreſſion of numbers, ſo that b ſhall be leſs than a , and c than b , and d than c , and ſo on of the following numbers, *ad infinitum*.

And 2dly, let theſe numbers be ſo related to each other, that they not only ſhall form a decreasing progreſſion themſelves, but that their differences, $a-b, b-c, c-d, d-e, e-f, f-g, g-h, \&c.$ ſhall alſo form a decreasing progreſſion, ſo that $b-c$ ſhall be leſs than $a-b$, and $c-d$ than $b-c$, and $d-e$ than $c-d$, and ſo on of the following differences; and likewise, that the differences of theſe differences (which may be called *the ſecond differences* of the original numbers $a, b, c, d, e, f, g, h, \&c.$ ſhall form a decreasing progreſſion; and that the differences of thoſe ſecond differences, or *the third differences* of the original

numbers $a, b, c, d, e, f, g, h, \&c.$ shall also form a decreasing progression; and in like manner, that the differences of the said third differences, or the *fourth differences*, of the original numbers $a, b, c, d, e, f, g, h, \&c.$ and the fifth and sixth differences, and all higher differences, of the same numbers, shall also form decreasing progressions.

And 3dly, let x be a quantity of any magnitude not greater than unity.

Upon these suppositions the value of the infinite series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c.$ (in which the second, fourth, sixth, and eighth, and every following even term, is marked with the sign $-$, or is to be subtracted from that which immediately precedes it) may be determined in the following manner.

Art. 2. Compute the first, second, third, fourth, and other subsequent differences of the co-efficients of the powers of x in this series, that is, of the numbers $b, c, d, e, f, g, h, \&c.$ as far as shall be convenient. These differences will be as follows.

First differences, $b - c, c - d, d - e, e - f, f - g, g - h, \&c.$

Second differences,

$$b - c - \overline{c - d}, c - d - \overline{d - e}, d - e - \overline{e - f}, e - f - \overline{f - g}, f - g - \overline{g - h}, \&c.$$

$$\text{or, } b - 2c + d, c - 2d + e, d - 2e + f, e - 2f + g, f - 2g + h, \&c.$$

Third differences,

$$b - 2c + d - \overline{c - 2d + e}, c - 2d + e - \overline{d - 2e + f}, d - 2e + f - \overline{e - 2f + g},$$

$$e - 2f + g - \overline{f - 2g + h}, \&c.$$

or,

or, $b-3e+3d-e$, $c-3d+3e-f$, $d-3e+3f-g$, $e-3f+3g-h$, &c.

Fourth differences, $b-3e+3d-e$ $\overbrace{[c-3d+3e-f]}$,

$c-3d+3e-f$ $\overbrace{[d-3e+3f-g]}$,

$d-3e+3f-g$ $\overbrace{[e-3f+3g-h]}$, &c.

or, $b-4e+6d-4e+f$, $c-4d+6e-4f+g$, $d-4e+6f-4g+h$, &c.

Fifth differences, $b-4e+6d-4e+f$ $\overbrace{[c-4d+6e-4f+g]}$,

$c-4d+6e-4f+g$ $\overbrace{[d-4e+6f-4g+h]}$, &c.

or, $b-5e+10d-10e+5f-g$, $c-5d+10e-10f+5g-h$, &c.

Sixth differences,

$b-5e+10d-10e+5f-g$ $\overbrace{[c-5d+10e-10f+5g-h]}$, &c.

or, $b-6e+15d-20e+15f-6g+h$, &c.

Let the first difference of the first order, to wit,

$b-a$, be called D^I ;

and the first difference of the second order, to wit,

$b-2c+d$, be called D^{II} ;

and the first difference of the third order, to wit,

$b-3c+3d-e$, be called D^{III} ;

and the first difference of the fourth order, to wit,

$b-4c+6d-4e+f$, be called D^{IV} ;

and the first difference of the fifth order, to wit,

$b-5c+10d-10e+5f-g$, be called D^V ;

and the first difference of the sixth order, to wit,

$b-6c+15d-20e+15f-6g+h$, be called D^{VI} ;

and in like manner let the first differences of the seventh,

eighth, ninth, and tenth, and every following order of

differences be denoted by D^{VII} , D^{VIII} , D^{IX} , D^X , &c. that is,

by the capital letter D , with a Roman numeral figure

annexed

annexed to it, expressing the order of differences to which it belongs.

These things being supposed, the aforefaid infinite series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c.$ will be equal to the following differential series, to wit,

$$a - \frac{bx}{1+x} - \frac{D^I xx}{1+x|^2} - \frac{D^{II} x^3}{1+x|^3} - \frac{D^{III} x^4}{1+x|^4} - \frac{D^{IV} x^5}{1+x|^5} - \frac{D^V x^6}{1+x|^6} - \frac{D^{VI} x^7}{1+x|^7} - \&c.;$$

in which series all the terms after the first term a are marked with the sign $-$, or are to be subtracted from that term.

Art. 3. If we insert the differences themselves instead of D^I , D^{II} , D^{III} , D^{IV} , D^V , &c. in the foregoing differential series (which it may perhaps sometimes be convenient to do) that series will be as follows: $a - \frac{bx}{1+x} - \sqrt{b-c} \times \frac{x^2}{1+x|^2}$

$$- \sqrt{b-2c+d} \times \frac{x^3}{1+x|^3} - \sqrt{b-3c+3d-e} \times \frac{x^4}{1+x|^4}$$

$$- \sqrt{b-4c+6d-4e+f} \times \frac{x^5}{1+x|^5}$$

$$- \sqrt{b-5c+10d-10e+5f-g} \times \frac{x^6}{1+x|^6}$$

$$- \sqrt{b-6c+15d-20e+15f-6g+bx} \times \frac{x^7}{1+x|^7} - \&c. \text{ ad infinitum.}$$

Of the convergency of the foregoing differential series.

Art. 4. The foregoing differential series will always converge with a considerable degree of swiftness, so
that

that fix or eight of its terms will give the value of the whole (and consequently that of the original series $a-bx+cx^2-dx^3+ex^4-fx^5+gx^6-hx^7+\&c.$ to which it is equal) exact to several places of figures, even in the most difficult cases: for if x is $= 1$ (which is its greatest possible magnitude) $1+x$ will be $= 1+1$ or 2 , and consequently $\overline{1+x}^2$, $\overline{1+x}^3$, $\overline{1+x}^4$, $\overline{1+x}^5$, and the following powers of $1+x$, will be equal to $4, 8, 16, 32$, and the following powers of 2 ; and the powers of the fraction $\frac{x}{1+x}$ will be equal to the powers of $\frac{1}{2}$. Therefore the series

$$a - \frac{bx}{1+x} - \frac{D^1 xx}{1+x^2} - \frac{D^{II} x^3}{1+x^3} - \frac{D^{III} x^4}{1+x^4} - \frac{D^{IV} x^5}{1+x^5} - \frac{D^V x^6}{1+x^6} - \frac{D^{VI} x^7}{1+x^7} - \&c. \text{ will}$$

$$\text{in this case be } = \text{ to } a - \frac{b}{2} - \frac{D^1}{4} - \frac{D^{II}}{8} - \frac{D^{III}}{16} - \frac{D^{IV}}{32} - \frac{D^V}{64} - \frac{D^{VI}}{128} - \&c.$$

the terms of which decrease in a greater proportion than that of 1 to 2 , because the numerators $a, b, D^1, D^{II}, D^{III}, D^{IV}, D^V, D^{VI}, \&c.$ form a decreasing progression, and the denominators increase in the proportion of 2 to 1 .

Of the investigation of the foregoing differential series.

Art. 5. The foregoing differential series was investigated by first, supposing the original series $a-bx+cx^2-dx^3+ex^4-fx^5+gx^6-hx^7+\&c.$ to be equal to another series whose terms should involve the same powers of x as the former, but in which every power of x should be multiplied

multiplied into the same power of the fraction $\frac{1}{1+x}$, in order to accelerate their convergency, and then inquiring what would be the co-efficients of the terms of such a series, if such a series is possible, and what would be the signs to be prefixed to them, or in what manner they would be connected with the first term, whether by addition or subtraction. In order to this inquiry, I denoted the unknown co-efficients of the assumed series by the capital letters P, Q, R, S, T, V, &c. and wrote down the terms of it near each other, without prefixing to them either of the signs + and -, but separated them from each other only by a comma; so that the fundamental equation, from which I derived the differential series above-mentioned, was as follows: $a-bx+cx^2-dx^3+ex^4-fx^5+gx^6-hx^7+\&c.$ is = P, $\frac{Qx}{1+x}$, $\frac{Rxx}{1+x^2}$, $\frac{Sx^3}{1+x^3}$, $\frac{Tx^4}{1+x^4}$, $\frac{Vx^5}{1+x^5}$, &c. By necessary deductions from this equation it appeared that P would be equal to a ; and that all the following terms, of the assumed series, to wit, $\frac{Qx}{1+x}$, $\frac{Rxx}{1+x^2}$, $\frac{Sx^3}{1+x^3}$, $\frac{Tx^4}{1+x^4}$, $\frac{Vx^5}{1+x^5}$, &c. must be subtracted from the first term P, or a ; and that Q would be equal to $b-c$, or D^I ; and $R=b-c-\sqrt{c-d}$, or $b-2c+d$, or D^{II} ; and $S=b-2c+d-\sqrt{c-2d+e}$, or $b-3c+3d-e$, or D^{III} ; and

T=

$\tau = b - 4c + 6d - 4e + f$, or D^{IV} ; and $v = b - 5c + 10d - 10e + 5f - g$, or D^V ; and so on of the following co-efficients, to wit, that every new co-efficient of the assumed series is equal to the first difference of the next order of the differences derived from the original co-efficients b, c, d, e, f, g, h , &c. And from hence I concluded that the series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c.$ was equal to the series,

$$a - \frac{bx}{1+x} - \frac{D^I xx}{(1+x)^2} - \frac{D^{II} x^3}{(1+x)^3} - \frac{D^{III} x^4}{(1+x)^4} - \frac{D^{IV} x^5}{(1+x)^5} - \frac{D^V x^6}{(1+x)^6} - \frac{D^{VI} x^7}{(1+x)^7} - \&c.$$

Art. 6. The thought of supposing the original series $a - bx + cxx - dx^3 + ex^4 - fx^5 + \&c.$ to be equal to the series $P, \frac{Qx}{1+x}, \frac{Rxx}{(1+x)^2}, \frac{Sx^3}{(1+x)^3}, \frac{T x^4}{(1+x)^4}, \frac{V x^5}{(1+x)^5}, \&c.$ containing the powers of x multiplied into the same powers of the fraction $\frac{1}{1+x}$ in order to accelerate their convergency, occurred to me in consequence of reading the late Mr. THOMAS SIMPSON'S Mathematical Differtations, p. 62, 63. concerning the summation of serieses, in which he makes a supposition of a similar kind. Yet there seems to be a considerable difference between his proposition and that which is the subject of these pages; for he seems to suppose his quantities p, q, r, s, t , &c. (which answer to a, b, c, d, e , &c. in the notation made use of in the above serieses) to form an increasing progression of terms, and accordingly subtracts p from q , and q from r ,

and r from s , and s from t , and so on; and he seems also to suppose the differences $q-p$, $r-q$, $s-r$, $t-s$, &c. to form an increasing progression, and every subsequent order of differences to form likewise an increasing progression, and accordingly subtracts $q-p$ from $r-q$, and $r-q$ from $s-r$, and $s-r$ from $t-s$, and so on; whereas in the foregoing series $a-bx+cx^2-dx^3+ex^4-fx^5+gx^6-hx^7+\&c.$ the numbers a, b, c, d, e, f, g, h , &c. are supposed to form a decreasing progression of terms, as they are most commonly found to do in the serieses that occur in the solution of mathematical or philosophical problems.

Examples of the usefulness of the foregoing differential serieses in finding the values of infinite serieses whose terms decrease very slowly.

Computations of the lengths of circular arcs by means of infinite serieses derived from their tangents.

Art. 7. It is well known, that if r be put for the radius of a circle, and t for the tangent of any arch in it that is not greater than 45° , the magnitude of the arch whose tangent is t will be expressed by the infinite series

$$t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.$$

This series converges with great swiftness when the tangent is

much less than the radius; but when the tangent is nearly equal to the radius, it converges exceeding slowly; and when it is quite equal to the radius, or the arch is equal to 45° , the decrease of the terms is so slow as to make the computation of it in the common way, by computing the value of its initial terms, absolutely impracticable. For Sir ISAAC NEWTON has observed concerning this series in that extreme case (which then becomes equal to $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} + \&c.$) and another series that is almost as slow as this, that to exhibit its value exact to twenty decimal places of figures, there would be occasion for no less than five thousand millions of its terms, to compute which would take up above a thousand years. See Sir ISAAC NEWTON's second letter to Mr. OLDENBURGH, dated October 24, 1676, in the *Commercium Epistolicum*, p. 159. In these cases therefore it will be convenient to make use of some artifice to discover the value of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^3} - \frac{t^7}{7r^5} + \frac{t^9}{9r^7} - \frac{t^{11}}{11r^9} + \frac{t^{13}}{13r^{11}} - \frac{t^{15}}{15r^{13}} + \&c.$; and we shall find the application of the differential series above-mentioned to be a very proper artifice for this purpose.

Art. 8. In order to make this application, we must consider the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^3} - \frac{t^7}{7r^5} + \frac{t^9}{9r^7} - \frac{t^{11}}{11r^9} + \frac{t^{13}}{13r^{11}} - \frac{t^{15}}{15r^{13}} + \&c.$ as being the product of the multiplication of t into the

series $1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \&c.$ and must substitute x instead of $\frac{t^2}{r^2}$ in the terms of this last series, by which means it will be converted into the series

$1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ This series is of the same form with the original series above-mentioned, $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c.$ the numeral co-efficients $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \&c.$ of the powers of x in the former series answering to the literal or general co-efficients $a, b, c, d, e, f, g, h, \&c.$ of the same powers in the latter series. And these numeral co-efficients evidently form a decreasing progression, as the co-efficients $a, b, c, d, e, f, g, h, \&c.$ are supposed to do; and we shall find, upon examination, that the differences of these numeral co-efficients, of the several successive orders, also constitute decreasing progressions, as the several successive orders of differences of the co-efficients $a, b, c, d, e, f, g, h, \&c.$ are supposed to do. Consequently the series

$1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ will be equal to the differential series

$$a - \frac{bx}{1+x} - \frac{D^I xx}{1+x|^2} - \frac{D^{II} x^3}{1+x|^3} - \frac{D^{III} x^4}{1+x|^4} - \frac{D^{IV} x^5}{1+x|^5} - \frac{D^V x^6}{1+x|^6} - \frac{D^{VI} x^7}{1+x|^7} - \&c. \text{ if}$$

we suppose the letters $a, b, c, d, e, f, g, h, \&c.$ to be equal to the numbers $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \&c.$ and $D^I, D^{II}, D^{III}, D^{IV}, D^V, D^{VI}, \&c.$ to be the first differences of

the feveral orders of differences of those numbers, beginning from the second term $\frac{1}{3}$. Now the values of these numbers, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}$, &c. and of their differences of the feveral fucceffive orders, beginning from the second term $\frac{1}{3}$, will, when expreffed in decimal fractions, be as follows:

$$1 \text{ is } = 1.000,000,000,000;$$

$$\frac{1}{3} = .333,333,333,333;$$

$$\frac{1}{5} = .200,000,000,000;$$

$$\frac{1}{7} = .142,857,142,857;$$

$$\frac{1}{9} = .111,111,111,111;$$

$$\frac{1}{11} = .090,909,090,909;$$

$$\frac{1}{13} = .076,923,076,923;$$

$$\frac{1}{15} = .066,666,666,666.$$

The differences of these numbers, beginning from the second term, .333,333,333,33, are as follows:

First differences.

Second differences.

$$.133,333,333,333;$$

$$.076,190,476,190;$$

$$.057,142,857,143;$$

$$.025,396,825,397;$$

$$.031,746,031,746;$$

$$.011,544,011,544;$$

$$.020,202,020,202;$$

$$.006,216,006,216;$$

$$.013,986,013,986;$$

$$.003,729,003,729;$$

$$.010,256,410,257;$$

$$\&c.$$

&c.

Third

Third differences.

.050,793,650,793;
 .013,852,813,853;
 .005,328,005,328;
 .002,486,402,487;
 &c.

Fourth differences.

.036,940,836,940;
 .008,524,808,525;
 .002,841,602,841;
 &c.

Fifth differences.

.028,416,028,415;
 .005,683,205,684;
 &c.

Sixth differences.

.022,732,822,731;
 &c.

Therefore D^I is = .133,333,333,333; D^{II} = .076,190,476,190; D^{III} = .050,793,650,793; D^{IV} = .036,940,836,940; D^V = .028,416,028,415; D^{VI} = .022,732,822,731.

Therefore the series $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ is
 equal to the series

$$\begin{aligned}
 &1 - .333,333,333,333, \times \frac{x}{1+x} \\
 &- .133,333,333,333, \times \frac{x^2}{1+x^2} \\
 &- .076,190,476,190, \times \frac{x^3}{1+x^3} \\
 &- .050,793,650,793, \times \frac{x^4}{1+x^4}
 \end{aligned}$$

- .036,

$$- .036,940,836,940, \times \frac{x^5}{1+x^2}$$

$$- .028,416,028,415, \times \frac{x^6}{1+x^2}$$

$$- .022,732,822,731, \times \frac{x^7}{1+x^2}$$

- &c.; and consequently the product of this latter series into the tangent t will be equal to the product of the former series $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \text{&c.}$ into the same quantity, that is, to the product of the series $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \text{&c.}$ into the tangent t , or to the original series

$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \text{&c.}$ which expresses the magnitude of the arch of which t is the tangent.

Computation of an arch of 30 degrees.

Art. 9. Now let t be the tangent of 30° , which is $= r \times \frac{1}{\sqrt{3}}$. Then will tt be $= \frac{rr}{3}$; and $\frac{tt}{rr}$, or x , $= \frac{rr}{3rr}$, or $\frac{1}{3}$. Therefore $1+x$ will be $= 1 + \frac{1}{3} = \frac{3}{3} + \frac{1}{3} = \frac{4}{3}$, and $\frac{x}{1+x}$ will be $= \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$. Therefore $\frac{xx}{1+x^2}$ will be $= \frac{1}{16}$, and $\frac{x^3}{1+x^2}$ $= \frac{1}{64}$, and $\frac{x^4}{1+x^2} = \frac{1}{256}$, and $x^5 = \frac{1}{1024}$, and $x^6 = \frac{1}{4096}$, and $x^7 = \frac{1}{16384}$. Consequently the differential series will in this

case

case be equal to

$$\begin{aligned}
 1 - & .333,333,333,333, \times \frac{1}{4} \\
 - & .133,333,333,333, \times \frac{1}{16} \\
 - & .076,190,476,190, \times \frac{1}{64} \\
 - & .050,793,650,793, \times \frac{1}{256} \\
 - & .036,940,836,940, \times \frac{1}{1024} \\
 - & .028,416,028,415, \times \frac{1}{4096} \\
 - & .022,732,822,731, \times \frac{1}{16384} \\
 - & \&c. = 1 - .083,333,333,333, \\
 & \quad - .008,333,333,333, \\
 & \quad - .001,190,476,190, \\
 & \quad - .000,198,412,698, \\
 & \quad - .000,036,075,036, \\
 & \quad - .000,006,937,506, \\
 & \quad - .000,001,387,501, \\
 & \quad - \&c.
 \end{aligned}$$

$$= 1 - .093,099,955,597, = 0.906,900,044,403.$$

Therefore the series $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ or

$1 - \frac{11}{3rr} + \frac{11^2}{5r^4} - \frac{11^3}{7r^6} + \frac{11^4}{9r^8} - \frac{11^5}{11r^{10}} + \frac{11^6}{13r^{12}} - \frac{11^7}{15r^{14}} + \&c.$ is in this case
 $= 0.906,900,044,403$, or (neglecting the latter figures
 after the sixth place of figures, because we are sure they

are

are not exact) 0.906,900. Therefore the product of the series $1 - \frac{t^2}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \&c.$ into the tangent t is equal to 0.906,900, $\times t = 0.906,900 \times r \times \frac{1}{\sqrt{3}}$

$= 0.906,900, \times r \times \frac{1}{1.732,050,8} = 0.906,900 \times r$
 $\times .577,350,2 = 0.523,598,8 \times r$; that is, the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^5} - \frac{t^7}{7r^7} + \frac{t^9}{9r^9} - \frac{t^{11}}{11r^{11}} + \frac{t^{13}}{13r^{13}} - \frac{t^{15}}{15r^{15}} + \&c.$ (which expresses the magnitude of the arch of which t is the tangent) is in this case $= 0.523,598,8 \times r$, or an arch of 30° is equal to $0.523,598,8 \times r$.

Art. 10. This value of an arch of 30° is exact in the six first places of figures, and errs only an unit in the seventh figure, which should be a 7 instead of an 8, the more exact value of that arch being 0.523,598,775,598, &c. And thus by the help of only eight terms of the differential series

$a - \frac{bx}{1+x} - \frac{D^1xx}{1+x^2} - \frac{D^{II}x^3}{1+x^3} - \frac{D^{III}x^4}{1+x^4} - \frac{D^{IV}x^5}{1+x^5} - \frac{D^Vx^6}{1+x^6} - \frac{D^{VI}x^7}{1+x^7} - \&c.$ we have obtained the value of the series

$t - \frac{t^3}{3rr} + \frac{t^5}{5r^5} - \frac{t^7}{7r^7} + \frac{t^9}{9r^9} - \frac{t^{11}}{11r^{11}} + \frac{t^{13}}{13r^{13}} - \frac{t^{15}}{15r^{15}} + \&c.$ in the case of an arch of 30 degrees, exact to six places of figures. This degree of exactness is the same with that which we should attain by computing twelve terms of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^5} - \frac{t^7}{7r^7} + \frac{t^9}{9r^9} - \frac{t^{11}}{11r^{11}} + \frac{t^{13}}{13r^{13}} - \frac{t^{15}}{15r^{15}} + \&c.$ itself, as will appear from the following calculation.

Art. 11. The series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}}$
 $- \frac{t^{15}}{15r^{14}} + \frac{t^{17}}{17r^{16}} - \frac{t^{19}}{19r^{18}} + \frac{t^{21}}{21r^{20}} - \frac{t^{23}}{23r^{22}} + \&c.$ is $= t \times$ the series

$1 - \frac{t^2}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \frac{t^{16}}{17r^{16}} - \frac{t^{18}}{19r^{18}} + \frac{t^{20}}{21r^{20}}$
 $- \frac{t^{22}}{23r^{22}} + \&c. =$, in the case of an arch of 30° , to $r \times \frac{1}{\sqrt{3}}$ into
 the series $1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 9} - \frac{1}{7 \times 27} + \frac{1}{9 \times 81} - \frac{1}{11 \times 243} + \frac{1}{13 \times 729}$
 $- \frac{1}{15 \times 2187} + \frac{1}{17 \times 6561} - \frac{1}{19 \times 19683} + \frac{1}{21 \times 59049} - \frac{1}{23 \times 177,147} + \&c.$

$= r \times \frac{1}{\sqrt{3}} \times$ the series 1.000,000,000,000, $-\frac{.333,333,333,333}{3}$

$+ \frac{.111,111,111,111}{5} - \frac{.037,037,037,037}{7}$

$+ \frac{.012,345,679,012}{9} - \frac{.004,115,226,337}{11}$

$+ \frac{.001,371,742,112}{13} - \frac{.000,457,247,370}{15}$

$+ \frac{.000,152,415,790}{17} - \frac{.000,050,805,263}{19}$

$+ \frac{.000,016,935,087}{21} - \frac{.000,005,645,029}{23}$

$+ \&c.$

$= r \times \frac{1}{\sqrt{3}} \times$ the series

1.000,000,000,000, $- .111,111,111,111,$

$+ .022,222,222,222, - .005,291,005,291,$

$+ .001,371,742,112, - .000,374,111,485,$

$+ .000,105,518,624, - .000,030,483,158,$

$+ .000,008,965,634, - .000,002,673,961,$

$+ .000,000,806,432, - .000,000,245,436,$

$+ \&c.$

I

=

$$= r \times \frac{1}{\sqrt{3}} \times \sqrt{1.023,709,255,024, - .116,809,630,442,} \\ + \&c.$$

$$= r \times \frac{1}{\sqrt{3}} \times .906,899,624,582,$$

$$= r \times \frac{1}{1.732,050,8} \times .906,899,624,582,$$

$= r \times .577,350,2 \times .906,899,624,582$, = (if we neglect the fix latter figures of 906,899,624,582, which we know to be not exact) $r \times .577,350,2 \times 906,899$, $= r \times .523,598,319,029,8$; of which the first fix figures .523,598, are exact.

Computation of an arch of 45 degrees.

Art. 12. Now let the tangent t be equal to the radius r , or the arch (whose magnitude is expressed by the series $t - \frac{t^3}{3r} + \frac{t^5}{5r^3} - \frac{t^7}{7r^5} + \frac{t^9}{9r^7} - \frac{t^{11}}{11r^9} + \frac{t^{13}}{13r^{11}} - \frac{t^{15}}{15r^{13}} + \&c.$) be an arch of 45° . This series will, in this case, become equal to $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} + \&c.$ of which the first eight terms will give the value of the whole exact to only one figure, as will appear by the following computation. These terms are equal to $r \times$ the eight terms

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15}, \text{ that is, to } r \times \text{the eight terms}$$

$$1.000,000,000,000, - .333,333,333,333,$$

$$+ .200,000,000,000, - .142,857,142,857,$$

$$+ .111,111,111,111, - .090,909,090,909,$$

$$+ .076,923,076,923, - .066,666,666,666,$$

$= r \times \sqrt{1.388,034,187,034, - .633,766,233,765,}$
 $= r \times .754,267,943,269;$ which agrees with the value
 of the whole series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \&c.$ only in the
 highest figure 7, the more exact value of that series
 being .785,398,163,397, &c. But, if we compute eight
 terms of the differential series which is equal to the
 series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \&c.$, we shall thereby obtain
 its value exact to three places of figures; which is as
 great a degree of exactness as would be attained by
 computing about five hundred terms of the series
 $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} + \&c.$ itself. The computation
 of the eight first terms of the said differential series is as
 follows.

Art. 13. Since t is in this case $= r$, tt will be $= rr$,
 and consequently $\frac{tt}{rr}$, or x , will be $= 1$. Therefore xx , x^3 ,
 x^4 , x^5 , and all the other powers of x , will in this case be
 equal to 1, and $1+x$ will be equal to $1+1$, or 2, and the
 powers of $1+x$ to the powers of 2. Therefore the frac-
 tion $\frac{x}{1+x}$ and its powers will be equal in this case to the
 fraction $\frac{1}{2}$ and its powers. Therefore the general differ-
 ential series in art. 8. to wit,

$$\begin{aligned}
 1 &- .333,333,333,333, \times \frac{x}{1+x} \\
 &- .133,333,333,333, \times \frac{x^2}{1+x^2} \\
 &- .076,190,476,190, \times \frac{x^3}{1+x^3} \\
 &- .050,793,650,793, \times \frac{x^4}{1+x^4} \\
 &- .036,940,836,940, \times \frac{x^5}{1+x^5} \\
 &- .028,416,028,415, \times \frac{x^6}{1+x^6} \\
 &- .022,732,822,731, \times \frac{x^7}{1+x^7}
 \end{aligned}$$

- &c. will become in this case equal to

$$\begin{aligned}
 1 &- .333,333,333,333, \times \frac{1}{2} \\
 &- .133,333,333,333, \times \frac{1}{4} \\
 &- .076,190,476,190, \times \frac{1}{8} \\
 &- .050,793,650,793, \times \frac{1}{16} \\
 &- .036,940,836,940, \times \frac{1}{32} \\
 &- .028,416,028,415, \times \frac{1}{64} \\
 &- .022,732,822,731, \times \frac{1}{128} \\
 &- \&c. = 1 - .166,666,666,666, \\
 &\quad - .033,333,333,333, \\
 &\quad - .009,523,809,523,
 \end{aligned}$$

$$= .003,174,603,174,$$

$$= .001,154,401,154,$$

$$= .000,444,000,443,$$

$$= .000,177,600,177,$$

$$= 8\&c.$$

$$= 1 - .214,474,414,470, = .785,525,585,530.$$

Therefore the series $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ or

$$1 - \frac{t}{3r} + \frac{t^2}{5r^2} - \frac{t^3}{7r^3} + \frac{t^4}{9r^4} - \frac{t^5}{11r^5} + \frac{t^6}{13r^6} - \frac{t^7}{15r^7} + \&c. \text{ is equal to}$$

$.785,525,585,530$; and consequently the series

$$t - \frac{t^3}{3r} + \frac{t^5}{5r^2} - \frac{t^7}{7r^3} + \frac{t^9}{9r^4} - \frac{t^{11}}{11r^5} + \frac{t^{13}}{13r^6} - \frac{t^{15}}{15r^7} + \&c. \text{ is in this case}$$

$= t \times .785,525,585,530, = r \times .785,525,585,530$; that is, the length of an arch of 45° , in a circle whose radius is r , is $= r \times .785,525,585,530$; which number is true to three places of figures, the more exact value of that arch being $r \times .785,398,163,397, \&c.$

Art. 14. It has been asserted in art. 12, that in order to obtain the value of the series

$$t - \frac{t^3}{3r} + \frac{t^5}{5r^2} - \frac{t^7}{7r^3} + \frac{t^9}{9r^4} - \frac{t^{11}}{11r^5} + \frac{t^{13}}{13r^6} - \frac{t^{15}}{15r^7} + \&c. \text{ exact to 3}$$

places of decimal figures by the mere computation of its terms, in the case of an arch of 45° , we must compute at least 500 of its terms. This may be proved in the following manner. The indexes of the powers of t in that series are the odd numbers 1, 3, 5, 7, 9, 11, 13, 15,

$\&c.$

&c. in their natural order; to which if we add an unit, the numbers thereby produced will be the even numbers 2, 4, 6, 8, 10, 12, 14, 16, &c. in their natural order, which are the doubles of the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c. Therefore the number of terms of that series from the beginning of it to any given term in it, including the said term, is always half the number that is produced by adding an unit to the index of t in the said term. Thus, if we take the term $\frac{t^{11}}{11r^{10}}$, and add 1 to 11, which is the index of the power of t in it, the sum will be 12, the half of which is 6, which is the number of terms in the series from the beginning of it to the term $\frac{t^{11}}{11r^{10}}$, including the said term, that term being the sixth term in the series. If therefore we take the term $\frac{t^{999}}{999r^{998}}$, and are desirous of knowing its place in the series, or the number of terms from the beginning of the series to that term inclusively, we must add 1 to the index of the power of t in its numerator, which will increase it to 1000; and half this sum, to wit, 500, will be the number of terms from the beginning of the series to the term $\frac{t^{999}}{999r^{998}}$ inclusively; or, in other words, this term will be the 500th term of the series. To arrive therefore at those terms of the series in which the indexes

dexes of the powers of t are greater than 999, or 1000, or in which the numeral co-efficients of the terms (which, by the law of this series, are equal to 1 divided by these indexes) are less than $\frac{1}{999}$ or $\frac{1}{1000}$, it is necessary to compute 500 of its terms. Now when t is $= r$, and consequently the literal parts of the terms of this series do not converge at all, it is evidently necessary to carry the computation as far as those terms in which the numeral co-efficients of the terms are less than $\frac{1}{999}$ or $\frac{1}{1000}$, in order to get the value of the series exact to the $\frac{1}{999}$ th or $\frac{1}{1000}$ th part of the radius r , or to the place of thousandths, or the third place of decimal figures. Therefore, when t is $= r$, or the arch is $= 45^\circ$, it is necessary to compute at least 500 terms of the series

$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.$, in order to obtain the value of it exact to three places of decimal figures, that is, to the same degree of exactness to which we attained in art. 13. by computing only eight terms of the above-mentioned differential series. *Q. E. D.*

Art. 15. But the best way of applying the aforesaid differential series to the investigation of the value of one of these very flow serieses, is to compute a moderate number of the first terms of the flow series in the common way, and then apply the differential series to the

computation of its remaining terms. The advantage of this method of proceeding will be manifest, if we apply it to the foregoing example of the series

$t = \frac{t^3}{3r^3} + \frac{t^5}{5r^5} + \frac{t^7}{7r^7} + \frac{t^9}{9r^9} + \frac{t^{11}}{11r^{11}} + \frac{t^{13}}{13r^{13}} + \frac{t^{15}}{15r^{15}} + \&c.$ in the case of an arch of 45° .

Compute therefore the first twelve terms of this series in the common way. These terms will be as follows :

$$t = r = r \times 1.000,000,000,000;$$

$$\frac{t^3}{3r^3} = \frac{r^3}{3r^3} = \frac{r}{3} = r \times .333,333,333,333;$$

$$\frac{t^5}{5r^5} = \frac{r^5}{5r^5} = \frac{r}{5} = r \times .200,000,000,000;$$

$$\frac{t^7}{7r^7} = \frac{r^7}{7r^7} = \frac{r}{7} = r \times .142,857,142,857;$$

$$\frac{t^9}{9r^9} = \frac{r^9}{9r^9} = \frac{r}{9} = r \times .111,111,111,111;$$

$$\frac{t^{11}}{11r^{11}} = \frac{r^{11}}{11r^{11}} = \frac{r}{11} = r \times .090,909,090,909;$$

$$\frac{t^{13}}{13r^{13}} = \frac{r^{13}}{13r^{13}} = \frac{r}{13} = r \times .076,923,076,923;$$

$$\frac{t^{15}}{15r^{15}} = \frac{r^{15}}{15r^{15}} = \frac{r}{15} = r \times .066,666,666,666;$$

$$\frac{t^{17}}{17r^{17}} = \frac{r^{17}}{17r^{17}} = \frac{r}{17} = r \times .058,823,529,411;$$

$$\frac{t^{19}}{19r^{19}} = \frac{r^{19}}{19r^{19}} = \frac{r}{19} = r \times .052,631,578,947;$$

$$\frac{t^{21}}{21r^{21}} = \frac{r^{21}}{21r^{21}} = \frac{r}{21} = r \times .047,619,047,619;$$

$$\frac{t^{23}}{23r^{23}} = \frac{r^{23}}{23r^{23}} = \frac{r}{23} = r \times .043,478,260,869.$$

Therefore the twelve terms $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$
 $+ \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \frac{t^{17}}{17r^{16}} - \frac{t^{19}}{19r^{18}} + \frac{t^{21}}{21r^{20}} - \frac{t^{23}}{23r^{22}}$ are

$$= \left\{ \begin{array}{l} r \times 1.000,000,000,000, -r \times .333,333,333,333, \\ +r \times .200,000,000,000, -r \times .142,857,142,857, \\ +r \times .111,111,111,111, -r \times .090,909,090,909, \\ +r \times .076,923,076,923, -r \times .066,666,666,666, \\ +r \times .058,823,529,411, -r \times .052,631,578,947, \\ +r \times .047,619,047,619, -r \times .043,478,260,869, \end{array} \right\}$$

$$= r \times 1.494,476,765,064, -r \times .729,876,073,581,$$

$$= r \times .764,600,691,483.$$

Having thus found the value of the first twelve terms of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \&c.$ to be $r \times .764,600,691,483$, we must apply the differential series to the discovery of the value of the remaining part of this series, which is the series

$\frac{t^{25}}{25r^{24}} - \frac{t^{27}}{27r^{26}} + \frac{t^{29}}{29r^{28}} - \frac{t^{31}}{31r^{30}} + \frac{t^{33}}{33r^{32}} - \frac{t^{35}}{35r^{34}} + \frac{t^{37}}{37r^{36}} - \frac{t^9}{39r^{38}} + \&c.$ *ad infinitum*. Now this series is equal to the product of $\frac{t^{25}}{r^{24}}$ into the series $\frac{1}{25} - \frac{t^2}{27rr} + \frac{t^4}{29r^4} - \frac{t^6}{31r^6} + \frac{t^8}{33r^8} - \frac{t^{10}}{35r^{10}} + \frac{t^{12}}{37r^{12}} - \frac{t^{14}}{39r^{14}} + \&c.$ or (putting x , as before, $= \frac{t^2}{rr}$) to the product of $\frac{t^{25}}{r^{24}}$ into the
serie

series $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} + \&c.$ which is of the same form with the series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c.$ Therefore, if we put $a = \frac{1}{25}$, $b = \frac{1}{27}$, $c = \frac{1}{29}$, $d = \frac{1}{31}$, $e = \frac{1}{33}$, $f = \frac{1}{35}$, $g = \frac{1}{37}$, $h = \frac{1}{39}$, and so on, and compute the differential series

$$a - \frac{bx}{1+x} - \frac{D^1xx}{1+x|^2} - \frac{D^{11}x^3}{1+x|^3} - \frac{D^{111}x^4}{1+x|^4} - \frac{D^{1111}x^5}{1+x|^5} - \frac{D^{11111}x^6}{1+x|^6} - \frac{D^{111111}x^7}{1+x|^7} - \&c.$$

thence resulting, the number thereby obtained will be

the value of the series $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} + \&c.$

or $\frac{1}{25} - \frac{11}{27rr} + \frac{1^4}{29r^4} - \frac{1^6}{31r^6} + \frac{1^8}{33r^8} - \frac{1^{10}}{35r^{10}} + \frac{1^{12}}{37r^{12}} - \frac{1^{14}}{39r^{14}} + \&c.$ This computation is as follows:

$$\text{Here } a \text{ is } = \frac{1}{25} = .040,000,000,000;$$

$$b = \frac{1}{27} = .037,037,037,037;$$

$$c = \frac{1}{29} = .034,482,758,620;$$

$$d = \frac{1}{31} = .032,258,064,516;$$

$$e = \frac{1}{33} = .030,303,030,303;$$

$$f = \frac{1}{35} = .028,571,428,571;$$

$$g = \frac{1}{37} = .027,027,027,027;$$

$$h = \frac{1}{39} = .025,641,025,641.$$

The differences of these numbers (beginning from the second number $\frac{1}{27}$, or .037,037,037,037,) are as follows:

First differences.

.002,554,278,416;
 .002,224,694,104;
 .001,955,034,413;
 .001,731,601,731;
 .001,544,491,544;
 .001,386,001,386.

Second differences.

.000,329,584,311;
 .000,269,659,891;
 .000,223,432,481;
 .000,187,200,187;
 .000,158,400,158.

Third differences.

.000,059,924,420;
 .000,046,227,409;
 .000,036,232,294;
 .000,028,800,028.

Fourth differences.

.000,013,697,010;
 .000,009,995,115;
 .000,007,432,265.

Fifth differences.

.000,003,701,894;
 .000,002,562,850.

Sixth differences.

.000,001,139,044.

Therefore D^I is = .002,554,278,416;

D^{II} = .000,329,584,311;

D^{III} = .000,059,924,420;

D^{IV} = .000,013,697,010;

D^V = .000,003,701,894;

D^{VI} = .000,001,139,044.

Consequently the differential series

$$a - \frac{bx}{1+x} - \frac{D^I x^2}{1+x^2} - \frac{D^{II} x^3}{1+x^3} - \frac{D^{III} x^4}{1+x^4} - \frac{D^{IV} x^5}{1+x^5} - \frac{D^V x^6}{1+x^6} - \frac{D^{VI} x^7}{1+x^7} - \&c. \text{ is to,}$$

.040,

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$$\begin{aligned}
 &.040,000,000,000, \\
 &- .037,037,037,037, \times \frac{x}{1+x} \\
 &- .002,554,278,416, \times \frac{x^2}{1+x^2} \\
 &- .000,329,584,311, \times \frac{x^3}{1+x^3} \\
 &- .000,059,924,420, \times \frac{x^4}{1+x^4} \\
 &- .000,013,697,010, \times \frac{x^5}{1+x^5} \\
 &- .000,003,701,894, \times \frac{x^6}{1+x^6} \\
 &- .000,001,139,044, \times \frac{x^7}{1+x^7} \\
 &- \&c.
 \end{aligned}$$

But, since t is in this case $= r$, $\frac{rt}{rr}$, or x , will be $= 1$, and consequently $\frac{x}{1+x}$ will be $= \frac{1}{1+1}$ or $\frac{1}{2}$. Therefore the foregoing differential series is in this case equal to

$$\begin{aligned}
 &.040,000,000,000, \\
 &- .037,037,037,037, \times \frac{1}{2} \\
 &- .002,554,278,416, \times \frac{1}{4} \\
 &- .000,329,584,311, \times \frac{1}{8} \\
 &- .000,059,924,420, \times \frac{1}{16} \\
 &- .000,013,697,010, \times \frac{1}{32}
 \end{aligned}$$

- .000,

$$- .000,003,701,894, \times \frac{1}{64}$$

$$- .000,001,139,044, \times \frac{1}{128}$$

$$\begin{aligned} - \&c. = .040,000,000,000, - .018,518,518,518, \\ &- .000,638,569,604, \\ &- .000,041,198,038, \\ &- .000,003,745,276, \\ &- .000,000,428,031, \\ &- .000,000,057,842, \\ &- .000,000,008,898, \\ &- \&c. \end{aligned}$$

$$= .040,000,000,000, - .019,202,526,207, - \&c.$$

$$= .020,797,473,793 - \&c.$$

Therefore the series $\frac{1}{25} - \frac{x}{27} + \frac{x^2}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} + \&c.$ or

$$\frac{1}{25} - \frac{11}{27rr} + \frac{1^4}{29r^4} - \frac{1^6}{31r^6} + \frac{1^8}{33r^8} - \frac{1^{10}}{35r^{10}} + \frac{1^{12}}{37r^{12}} - \frac{1^{14}}{39r^{14}} + \&c. \text{ is in this}$$

case $= .020,797,473,793$. Therefore the series

$$\frac{1^{25}}{25r^{25}} - \frac{1^{27}}{27r^{26}} + \frac{1^{29}}{29r^{28}} - \frac{1^{31}}{31r^{30}} + \frac{1^{33}}{33r^{32}} - \frac{1^{35}}{35r^{34}} + \frac{1^{37}}{37r^{36}} - \frac{1^{39}}{39r^{38}} + \&c. \text{ is in}$$

this case equal to $\frac{1^{25}}{r^{24}} \times .020,797,473,793$, that is, to

$$\frac{1^{25}}{r^{24}} \times .020,797,473,793, \text{ or } r \times .020,797,473,793; \text{ that}$$

is, the remainder of the infinite series

$$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \&c. \text{ after the first twelve}$$

terms, is $= r \times .020,797,473,793$. But we before found

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those first twelve terms to be $= r \times .764,600,691,483$.

Therefore the whole series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$
&c. *ad infinitum* is in this case $= r \times .764,600,691,483,$

$+ r \times .020,797,473,793, = r \times .785,398,165,276,$

which is true to eight places of figures, the more exact value of that series being $r \times .785,398,163,397,$ &c.; so that the value here found for this series, by the help of only eight terms of the differential series, differs from its true value by less than an unit in the eighth place of decimal figures, that is, by less than an hundred-millionth part of the radius r , which is a degree of exactness that could not have been attained by the mere computation of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$ &c. itself without computing fifty millions of its terms. There cannot be a stronger instance of the utility of that differential series.

Computation of the series which expresses the time of the descent of a pendulum through the arch of a circle.

Art. 16. As another example of the utility of the foregoing differential series in finding the value of a series that converges very slowly, I will now apply it to the series which expresses the time of descent of a heavy body through a circular arch of 90° , which decreases

creafes almoft as flowly as the above-mentioned feries $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} + \&c.$, which expreffes the magnitude of a circular arch of 45° in a circle whole radius is r .

Art. 17. If a heavy body, or a pendulum, be fupposed to defcend by the mere force of gravity through any arch of a circle not exceeding the arch of a quadrant, or 90° ; and the motion be fupposed to begin from a ftate of reft, and to continue till the bob of the pendulum, or the heavy body, comes to the loweft point of the circle; and the radius of the circle be called r , the perpendicular height, or verfed fine, of the arch through which the defcent is made, be called v , and the right fine of the fame arch be called s ; and π be put for the number 1.570,796,326,794, &c. which expreffes the femi-circumference of a circle whole diameter is called 1; and the time of the fall of a heavy body through the verfed fine v , or the perpendicular altitude of the arch through which the pendulum defcends, be denoted by v ; the time of the defcent of the pendulum through the faid circular arch, correfponding to the verfed fine or altitude v , to the loweft point of the circle, will be expreffed by the product of $\pi \times \frac{r}{s}$ into the feries

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \frac{1.1.3.3.5.5.7.7}{2.2.4.4.6.6.8.8} \times \frac{v^8}{s^8}$$

$-\frac{1.1.3.3.5.5.7.7.9.9}{2.2.4.4.6.6.8.8.10.10} \times \frac{v^{10}}{s^{10}} + \&c.$ in which the law of the continuation of the terms is very manifest, every new term being derived from the preceding term by multiplying it into the fraction $\frac{vv}{ss}$, and likewise into a numeral fraction, whose denominator is the square of the index of the powers of v and s in the new term, and whose numerator is the square of the odd number that is less than the said index by an unit.

Art. 18. Let the numeral co-efficients of the terms of this series be denoted by the capital letters of the alphabet, A, B, C, D, E, F, G, H, &c. in their natural order, so that A shall be equal to 1, and B shall be $= \frac{1.1}{2.2}$, and $C = \frac{1.1.3.3}{2.2.4.4}$, and $D = \frac{1.1.3.3.5.5}{2.2.4.4.6.6}$, and so on of the rest. And we shall have

$$B = \frac{1.1}{2.2} A, C = \frac{3.3}{4.4} B, D = \frac{5.5}{6.6} C, E = \frac{7.7}{8.8} D, F = \frac{9.9}{10.10} E, G = \frac{11.11}{12.12} F, H = \frac{13.13}{14.14} G, \text{ and so on; and consequently the series}$$

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \&c.$$

OR $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \frac{Gv^{12}}{s^{12}} - \frac{Hv^{14}}{s^{14}} + \&c.$ will be $A - \frac{1.1.A.vv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} + \frac{11.11.Fv^{12}}{12.12.s^{12}} - \frac{13.13.Gv^{14}}{14.14.s^{14}} + \&c.$ OR $A - \frac{1Avv}{4ss} + \frac{9Bv^4}{16s^4} - \frac{25Cv^6}{36s^6} + \frac{49Dv^8}{64s^8} - \frac{81Ev^{10}}{100s^{10}} + \frac{121Fv^{12}}{144s^{12}} - \frac{169Gv^{14}}{196s^{14}} + \&c.$ or, if we convert the co-efficients of the terms into decimal fractions,

$$\begin{aligned}
& 1 - .250,000,000,000, \times \frac{v^2}{s^2} + .140,625,000,000, \times \frac{v^4}{s^4} \\
& - .097,656,250,000, \times \frac{v^6}{s^6} + .074,768,066,406, \times \frac{v^8}{s^8} \\
& - .060,562,133,788, \times \frac{v^{10}}{s^{10}} + .050,889,015,196, \times \frac{v^{12}}{s^{12}} \\
& - .043,878,793,714, \times \frac{v^{14}}{s^{14}} + \&c.
\end{aligned}$$

The co-efficients of these terms decrease so slowly (especially after the first twelve or fourteen terms) that, when the versed sine v is very nearly equal to the right sine s (as is the case when the arch through which the heavy body descends is nearly equal to 90° , or the arch of a whole quadrant of a circle) it would be necessary to compute a vast number of the terms of the series in order to obtain its value exact to seven or eight places of figures; and, when v is quite equal to s (as is the case when the arch, through which the descent is made, is exactly equal to 90°) the computation of the value of the series to that degree of exactness in that direct manner becomes wholly impracticable. But by the help of the differential series above-mentioned its value may be found, even in this case, to that degree of exactness without much difficulty; more especially if we compute the first twelve terms of the series in the common way, and then apply the differential series to the investigation of the remaining part of it in the same manner as in the last example. This we shall now proceed to do.

Art. 19. The co-efficients of the first twelve terms of the series $A = \frac{1Avv}{4s^2} + \frac{9Bv^4}{16s^4} - \frac{25Cv^6}{36s^6} + \frac{49Dv^8}{64s^8} - \frac{81Ev^{10}}{100s^{10}} + \frac{121Fv^{12}}{144s^{12}} - \frac{169Gv^{14}}{196s^{14}} + \&c.$ are as follows:

$$A = I = 1.000,000,000,000;$$

$$B = \frac{I}{4} = .250,000,000,000;$$

$$C = \frac{9B}{16} = .140,625,000,000;$$

$$D = \frac{25C}{36} = .097,656,250,000;$$

$$E = \frac{49D}{64} = .074,768,066,406;$$

$$F = \frac{81E}{100} = .060,562,133,788;$$

$$G = \frac{121F}{144} = .050,889,015,196;$$

$$H = \frac{169G}{196} = .043,878,793,714;$$

$$I = \frac{225H}{256} = .038,565,346,037;$$

$$K = \frac{289I}{324} = .034,399,336,434;$$

$$L = \frac{361K}{400} = .031,045,401,131;$$

$$M = \frac{441L}{484} = .028,287,235,328.$$

But when v is $= s$, as it is in the case of an arch of 90° ,

$\frac{vv}{ss}$ and all its powers will be $= 1$, and the twelve terms

$$A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \frac{Gv^{12}}{s^{12}} - \frac{Hv^{14}}{s^{14}} + \frac{Iv^{16}}{s^{16}} - \frac{Kv^{18}}{s^{18}} + \frac{Lv^{20}}{s^{20}} - \frac{Mv^{22}}{s^{22}}$$

will be equal to their co-efficients $A-B+C-D+E-F+G-H+I-K+L-M$. Therefore in this case the first twelve terms of this series are

$$\begin{aligned}
 & 1.000,000,000,000, - .250,000,000,000, \\
 & + .140,625,000,000, - .097,656,250,000, \\
 & + .074,768,066,406, - .060,562,133,788, \\
 & + .050,889,015,196, - .043,878,793,714, \\
 & + .038,565,346,037, - .034,399,336,434, \\
 & + .031,045,401,131, - .028,287,235,328, \\
 & \text{which are} = 1.335,892,828,770, - .514,783,749,264, \\
 & = .821,109,079,506.
 \end{aligned}$$

Art. 20. The remaining part of this series is

$$\begin{aligned}
 & \frac{Nv^{24}}{s^{24}} - \frac{Ov^{26}}{s^{26}} + \frac{Pv^{28}}{s^{28}} - \frac{Qv^{30}}{s^{30}} + \frac{Rv^{32}}{s^{32}} - \frac{sv^{34}}{s^{34}} + \frac{Tv^{36}}{s^{36}} - \frac{Vv^{38}}{s^{38}} + \&c. \\
 \text{OR } & \frac{23.23.Mv^{24}}{24.24.s^{24}} - \frac{25.25.Nv^{26}}{26.26.s^{26}} + \frac{27.27.Ov^{28}}{28.28.s^{28}} - \frac{29.29.Pv^{30}}{30.30.s^{30}} + \frac{31.31.Qv^{32}}{32.32.s^{32}} \\
 & - \frac{33.33.Rv^{34}}{34.34.s^{34}} + \frac{35.35.sv^{36}}{36.36.s^{36}} - \frac{37.37.Tv^{38}}{38.38.s^{38}} + \&c. \text{ or} \\
 & .025,979,075,500, \times \frac{v^{24}}{s^{24}} - .024,019,115,661 \times \frac{v^{26}}{s^{26}} \\
 & + .022,334,101,169, \times \frac{v^{28}}{s^{28}} - .020,869,976,759 \times \frac{v^{30}}{s^{30}} \\
 & + .019,585,984,048, \times \frac{v^{32}}{s^{32}} - .018,450,810,232, \times \frac{v^{34}}{s^{34}} \\
 & + .017,440,001,955, \times \frac{v^{36}}{s^{36}} - .016,534,184,679, \times \frac{v^{38}}{s^{38}} \\
 & + \&c.; \text{ which is } = \frac{v^{24}}{s^{24}} \times \text{the series}
 \end{aligned}$$

$$.025,979,075,500, - .024,019,115,661, \frac{vv}{ss}$$

$$+ .022,$$

$$\begin{aligned}
 &+ .022,334,101,169, \frac{v^4}{j^4} - .020,869,976,759, \frac{v^5}{j^5} \\
 &+ .019,585,984,048, \frac{v^6}{j^6} - .018,450,810,232, \frac{v^{10}}{j^{10}} \\
 &+ .017,440,001,955, \frac{v^{12}}{j^{12}} - .016,534,184,679, \frac{v^{14}}{j^{14}} \\
 &+ \&c. \text{ or, if we substitute } x \text{ in this last series instead of } \\
 &\frac{vv}{jj}, = \frac{v^{14}}{j^{14}} \times \text{the series}
 \end{aligned}$$

$$\begin{aligned}
 &.025,979,075,500, - .024,019,115,661, x \\
 &+ .022,334,101,169, xx - .020,869,976,759, x^3 \\
 &+ .019,585,984,048, x^4 - .018,450,810,232, x^5 \\
 &+ .017,440,001,955, x^6 - .016,534,184,679, x^7 \\
 &+ \&c. \text{ Now the value of this last series may be disco-} \\
 &\text{vered by the application of the differential series}
 \end{aligned}$$

$$a - \frac{bx}{1+x} - \frac{D^1 xx}{1+x^2} - \frac{D^{11} x^3}{1+x^4} - \frac{D^{111} x^4}{1+x^8} - \frac{D^{1111} x^5}{1+x^{16}} - \frac{D^v x^6}{1+x^{32}} - \frac{D^{v1} x^7}{1+x^{64}} - \&c.$$

in the manner following:

Here a is = .025,979,075,500;

b = .024,019,115,661;

c = .022,334,101,169;

d = .020,869,976,759;

e = .019,585,984,048;

f = .018,450,810,232;

g = .017,440,001,955;

and h = .016,534,184,679.

Therefore

Therefore the differences of b, c, d, e, f, g , and h , of the several successive orders, are as follows:

First differences.

$$\begin{aligned} b - c & \text{ is } = .001,685,014,492; \\ c - d & = .001,464,124,410; \\ d - e & = .001,283,992,711; \\ e - f & = .001,135,173,816; \\ f - g & = .001,010,808,277; \\ g - h & = .000,905,817,276. \end{aligned}$$

Second differences.

$$\begin{aligned} & .000,220,890,082; \\ & .000,180,131,699; \\ & .000,148,818,895; \\ & .000,124,365,539; \\ & .000,104,991,001. \end{aligned}$$

Third differences.

$$\begin{aligned} & .000,040,758,383; \\ & .000,031,312,804; \\ & .000,024,453,356; \\ & .000,019,374,538. \end{aligned}$$

Fourth differences.

$$\begin{aligned} & .000,009,445,579; \\ & .000,006,859,448; \\ & .000,005,078,818. \end{aligned}$$

Fifth differences.

$$\begin{aligned} & .000,002,586,131; \\ & .000,001,780,630. \end{aligned}$$

Sixth differences.

$$.000,000,805,501.$$

Therefore D^I is $= .001,685,014,492;$

$$D^{II} = .000,220,890,082;$$

$$D^{III} = .000,040,758,383;$$

$$D^{IV} = .000,009,445,579;$$

$$D^V = .000,002,586,131;$$

$$D^{VI} = .000,000,805,501.$$

Consequently the differential series

$$a - \frac{bx}{1+x} - \frac{D^I x^2}{1+x^2} - \frac{D^{II} x^3}{1+x^3} - \frac{D^{III} x^4}{1+x^4} - \frac{D^{IV} x^5}{1+x^5} - \frac{D^V x^6}{1+x^6} - \frac{D^{VI} x^7}{1+x^7} - \&c. \text{ is to}$$

$$\begin{aligned}
 &.025,979,075,500, \\
 &- .024,019,115,661, \times \frac{x}{1+x} \\
 &- .001,685,014,492, \times \frac{x^2}{1+x^2} \\
 &- .000,220,890,082, \times \frac{x^3}{1+x^3} \\
 &- .000,040,758,383, \times \frac{x^4}{1+x^4} \\
 &- .000,009,445,579, \times \frac{x^5}{1+x^5} \\
 &- .000,002,586,131, \times \frac{x^6}{1+x^6} \\
 &- .000,000,805,501, \times \frac{x^7}{1+x^7} \\
 &- \&c.
 \end{aligned}$$

This is the general value of the said differential series, whatever may be the value of x , or $\frac{vv}{ss}$. But in the case here supposed of an arch of 90° , the verfed sine v is equal to the sine s ; and therefore $\frac{vv}{ss}$, or x , is = 1, and $\frac{x}{1+x} = \frac{1}{1+1}$, or $\frac{1}{2}$. Therefore the foregoing differential series is in this case equal to

$$\begin{aligned}
 &.025,979,075,500, \\
 &- .024,019,115,661, \times \frac{1}{2} \\
 &- .001,685,014,492, \times \frac{1}{4}
 \end{aligned}$$

- .000,

$$\begin{aligned}
& - .000,220,890,082, \times \frac{1}{8} \\
& - .000,040,758,383, \times \frac{1}{16} \\
& - .000,009,445,579, \times \frac{1}{32} \\
& - .000,002,586,131, \times \frac{1}{64} \\
& - .000,000,805,501, \times \frac{1}{128} \\
& - \&c. = .025,979,075,500 - .012,009,557,830, \\
& \quad - .000,421,253,623, \\
& \quad - .000,027,611,260, \\
& \quad - .000,002,547,398, \\
& \quad - .000,000,295,174, \\
& \quad - .000,000,040,408, \\
& \quad - .000,000,006,292, \\
& \quad - \&c. \\
& = .025,979,075,500, - .012,461,311,985, \&c. \\
& = .013,517,763,515, - \&c. \quad \text{Therefore the series} \\
& \quad a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c., \text{ or} \\
& \quad .025,979,075,500 - .024,019,115,661, x \\
& \quad + .022,334,101,169, x^2 - .020,869,976,759, x^3 \\
& \quad + .019,585,984,048, x^4 - .018,450,810,232, x^5 \\
& \quad + .017,440,001,955, x^6 - .016,534,184,679, x^7 \\
& + \&c. \text{ is in this case} = .013,517,763,515, - \&c. \\
& \text{Therefore } \frac{v^{24}}{x^{24}} \times \text{this last series, or } \frac{v^{24}}{x^{24}} \times \text{the series}
\end{aligned}$$

$$\begin{aligned}
 &.025,979,075,500, - .024,019,115,661, \frac{v^8}{s^8} \\
 &+ .022,334,101,169, \frac{v^4}{s^4} - .020,869,976,759, \frac{v^6}{s^6} \\
 &+ .019,585,984,048, \frac{v^2}{s^2} - .018,450,810,232, \frac{v^{10}}{s^{10}} \\
 &+ .017,440,001,955, \frac{v^{12}}{s^{12}} - .016,534,184,679, \frac{v^{14}}{s^{14}} \\
 &+ \&c. \text{ is in this case} = \frac{v^{24}}{s^{24}} \times .013,517,763,515, - \&c.; \\
 &\text{that is, the series}
 \end{aligned}$$

$$\begin{aligned}
 &.025,979,075,500, \frac{v^{24}}{s^{24}} - .024,019,115,661, \frac{v^{26}}{s^{26}} \\
 &+ .022,334,101,169, \frac{v^{28}}{s^{28}} - .020,869,976,759, \frac{v^{30}}{s^{30}} \\
 &+ .019,585,984,048, \frac{v^{32}}{s^{32}} - .018,450,810,232, \times \frac{v^{34}}{s^{34}} \\
 &+ .017,440,001,955, \frac{v^{36}}{s^{36}} - .016,534,184,679, \frac{v^{38}}{s^{38}} \\
 &+ \&c., \text{ or } \frac{Nv^{24}}{s^{24}} - \frac{Ov^{26}}{s^{26}} + \frac{Pv^{28}}{s^{28}} - \frac{Qv^{30}}{s^{30}} + \frac{Rv^{32}}{s^{32}} - \frac{Sv^{34}}{s^{34}} + \frac{Tv^{36}}{s^{36}} - \frac{Vv^{38}}{s^{38}} + \&c. \text{ is} \\
 &\text{in this case} = \frac{v^{24}}{s^{24}} \times .013,517,763,515, - \&c. = (\text{because} \\
 &v \text{ is in this case} = s, \text{ and consequently } \frac{v^{24}}{s^{24}} \text{ is} = 1)
 \end{aligned}$$

.013,517,763,515, - &c. But we before found the value of the first twelve terms of the series

$$\begin{aligned}
 &A - \frac{Bvv}{s^2} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \&c. \text{ to be in this case} = \\
 &.821,109,079,506. \text{ Therefore the value of the whole} \\
 &\text{series } A - \frac{Bvv}{s^2} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \&c. \text{ ad infinitum is in} \\
 &\text{this case} = .821,109,079,506, + .013,517,763,515,
 \end{aligned}$$

$-\&c. = 834,626,843,021, -\&c.$ of which the first eight figures .834,626,84 are exact, the error being in the ninth figure 3, which ought to be a 2 instead of a 3, as would have appeared if we had computed another term or two of the differential series.

Art. 21. Since the series $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}}$
 $+ \&c.$, or $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} - \&c.$
 is, in this case of an arch of 90° , $= .834,626,843, -\&c.$,
 or somewhat less than .834,626,843, the product of that
 series into $\pi \times \frac{rv}{s}$, will be $= \pi \times \frac{rv}{s} \times .834,626,843, -\&c. =$
 (because v is in this case $= s$) $\pi \times r \times .834,626,843, -\&c.$
 $= 1.570,796,326,794, \&c. \times r \times .834,626,843$
 $-\&c. = r \times 1.311,028,779, -\&c.$, or something less than
 $r \times 1.311,028,779$; which is exact to nine places of
 figures, the more exact value of this quantity being
 $r \times 1.311,028,777,146, \&c.$ as appears by a computa-
 tion made by Mr. STIRLING, in his admirable Treatise on
 the Summation of Serieses, p. 58.

Art. 22. This value of the product of $1.570,796,326,794, \&c. \times \frac{rv}{s}$ into the series

$1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} + \&c.$, found by
 the foregoing processes in this extreme and most difficult
 case, to wit, $1.311,028,779, \&c. \times r$, exceeds its true
 value,

value, 1.311,028,777,146, &c. $\times r$ by only .000,000, 002, $\times r$, or two thousand-millionth parts of the radius r ; which is indeed a most minute difference, and shews the great exactness and utility of this differential series.

Art. 23. Of the nine figures to which the number 1.311,028,779, found by the foregoing process, is exact, the last eight are owing to the differential series. For if we were to multiply the value of the first twelve terms only of the series $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \&c.$, or $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} + \&c.$, to wit, the number .821,109,079,506, into $\pi \times r$, or 1.570,796, 326,794, &c. $\times r$, the product would be only 1.289, &c. $\times r$, which is true to only one place of figures, the second figure being a 2 instead of a 3. This therefore is an eminent proof of the utility of the said differential series.

Art. 24. In an arch of 90° the versed sine is equal to the radius of the circle, that is, according to the foregoing notation, v is $= r$. Therefore by art. 17. together with the foregoing computation, it appears, that the time of the descent of a pendulum, or other heavy body (moving freely from a state of rest by the force of gravity only) through the arch of a whole quadrant of a circle is to the time of the fall through the cor-

respondent perpendicular altitude, or the radius, as 1.311,028,779, - &c. $\times r$ is to r , or as 1.311,028,779, - &c. is to 1.

Art. 25. Hence we may determine the proportion of the time of descent of a pendulum through an arch of 90° to the time of its descent through an infinitely small arch at the bottom of a quadrant, or rather (to speak correctly) to the limit of the time of descent through a very small but finite arch at the bottom of the quadrant, to which the said time continually approaches nearer and nearer as the said small arch is taken less and less, and to which it may be made to approach so nearly, by taking the said small arch sufficiently small, as to differ from it by less than any given quantity. For this latter time, or limit, is known to be to the time of the fall of a heavy body through half the length of the pendulum, or half the radius of the circle, as the semi-circumference of a circle is to its diameter, that is, as the number 1.570,796,326,794, &c. is to 1. But the time of the fall of a heavy body through half the radius of the circle is to the time of the fall through the whole radius as 1 to $\sqrt{2}$, or 1.414,213, &c. Therefore, *ex æquo*, the said limit of the time of descent of a pendulum through a very small arch of the circle at the bottom of the quadrant, is to the time of the fall of a heavy body through

the radius of the circle, or the whole length of the pendulum, as 1.570,796, &c. is to 1.414,213, &c. But we have seen in the last article that the time of the fall of a heavy body through the radius of the circle is to the time of descent of a pendulum through the arch of a whole quadrant as 1 to 1.311,028,779, - &c. Therefore the limit of the time of descent of a pendulum through a very small arch at the bottom of the quadrant is to the time of descent through the arch of the whole quadrant as 1.570,796,326, &c. \times 1 is to 1.414,213, &c. \times 1.311,028,779, - &c., or as 1.570,796,326, &c. is to 1.414,213, &c. \times 1.311,028,779, - &c., that is, by art. 21. as 1.570,796,326, &c. is to 1.414,213, &c. \times 1.570,796,326, &c. \times .834,626,843, - &c., or as 1 to 1.414,213, &c. \times .834,626,843, - &c., or as 1 to 1.180,340, &c., or, in smaller numbers, as 1 to 1.180, or as 1000 to 1180, or as 100 to 118, or as 50 to 59.

Art. 26. This proportion of the times of the descent of a pendulum through an infinitely small arch at the bottom of the quadrant, and through the arch of the whole quadrant, agrees pretty nearly with that assigned for them by Mr. HUYGENS in the preface to his admirable Treatise on Pendulum-clocks, or *De Horologio Oscillatorio*, which is that of 29 to 34. For 50 is to 59 as 29 is to 34.2, or $34\frac{1}{5}$; or (neglecting the fraction $\frac{1}{5}$) as 29 is to 34; Mr. HUYGENS
meaning

meaning, probably, in that place, not to express this proportion as accurately as he could, but only as nearly as it could be expressed by small whole numbers. However, the numbers 50 and 59 express this proportion rather more accurately than 29 and 34, and with pretty much the same degree of simplicity, and therefore, upon the whole, are somewhat to be preferred to them.

Art. 27. I have endeavoured to find another differential series, similar to that above described, for the purpose of investigating the value of an infinite series of this form, to wit, $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c.$ (in which all the terms are marked with the sign +, or are added to the first term a) when the co-efficients $b, c, d, e, f, g, h, \&c.$ decrease very slowly, and x is very nearly equal to 1, and the terms of the series decrease consequently so slowly as to make the summation of it in the common way, or by the mere computation and addition of its terms, almost impracticable; but my endeavours have not been attended with success. I may therefore, from my own experience, subscribe to the truth of what is asserted upon this subject by the very learned and ingenious Mr. JAMES STIRLING in his Treatise, intitled, *Summatio Serierum*, p. 17. to wit, that *Series quarum termini sunt per vices negativi et affirmativi, sunt magis tractabiles quam alteræ, ubi de Summatione agitur*; though at first sight one would be apt to imagine the reverse of this proposition to be true.